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Evolution of Orbits and Quantum Correlation Functions by Quadratic Hamiltonians

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Quantum systems with quadratic Hamiltonians are considered. Some results about the time evolution of homogeneous polynomials and of quantum correlation functions are given. The image of arbitrary orbit of Weyl–Heisenberg group under this time evolution is shown to be again an orbit of this group. For quantum free particle it is shown that its time evolution intersects arbitrary such orbit at most once. A result about existence of more orbits having the same dispersion of some quantum position is presented.

KEY WORDS: quadratic Hamiltonian; quantum correlation function; orbit of Weyl–Heisenberg group.
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1. INTRODUCTION

We shall consider orbits of irreducible unitary representations of Weyl–Heisenberg group (Želobenko and Stern, 1983) in projective Hilbert space (Marsden and Ratiu, 1999) given by Weyl operators (Weyl, 1931). These orbits are special cases of generalized coherent states which have many important applications in mathematical physics (Perelomov, 1987). Quite a recent application of generalized coherent states is the theory of "classical projections of quantum mechanics" given by Bóna (1986) (and also in Bóna (2000) where it is called "restricted flows"). In this context, if the group is chosen to be the Weyl–Heisenberg group, some results about the classical limit of quantum mechanics were achieved by Polakovič (1998, 2001a,b). A result of this type and some other results for systems with quadratic Hamiltonians are given in Polakovič (2000). The results given in the present paper arised as a natural continuation of the just mentioned work of the present author.

Quadratic Hamiltonians (in positions and momenta) are of great importance in both classical and quantum mechanics. (Simple examples are, the harmonic

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oscillator and the free particle.) In the whole paper, we shall consider quantum systems with quadratic Hamiltonians.

Monomials and homogenous polynomials in quantum positions and momenta are considered. From the fact that the commutator of the quadratic Hamiltonian and a monomial of degree m is a homogeneous polynomial of degree m (Lemma 1), it is deduced that the time evolution of a monomial in Heisenberg picture for a quadratic Hamiltonian is a homogenous polynomial of the same degree (Theorem 2). It follows that the time evolution of mean values of a monomial (Corollary 3) and also of arbitrary quantum correlation function (Corollary 4) can be explicitly computed if we know the mean values of certain monomials in the initial state (if, again, the Hamiltonian is quadratic).

It is then shown that the evolution operators (for quadratic Hamiltonians) and the usual Weyl operators in a sense "commute" (Theorem 5). It follows that the image of arbitrary orbit of Weyl–Heisenberg group under the time evolution of a quadratic Hamiltonian (for a fixed time t) is again an orbit of Weyl–Heisenberg group (Corollary 6).

Finally, the quadratic Hamiltonian is specified to be the Hamiltonian of quantum free particle. Using some auxiliary statements (Lemma 7, Lemma 8, Lemma 9) it is shown that if the initial state of the free particle is on an orbit of Weyl–Heisenberg group then the time evolution never intersects this orbit again (Theorem 10). A simple consequence is that for sufficiently large number there always exist two different orbits of Weyl–Heisenberg group such that the (constant) value of the dispersion of some quantum position is the given number for both orbits (Corollary 11).

2. PRELIMINARIES

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and $P(\mathcal{H})$ be the corresponding projective Hilbert space. If $\psi \in \mathcal{H}$ is a vector then $\underline{\psi} = P_\psi \in \mathcal{H}$ will denote the corresponding projector.

Let us denote by

$$x = (x_1, \dots, x_{2n}) = (q_1, \dots, q_n, p_1, \dots, p_n) \in \mathbb{R}^{2n}$$

arbitrary point in the "flat" phase space for a classical Hamiltonian system with n degrees of freedom. Now

$$X = (X_1, \dots, X_{2n}) = (Q_1, \dots, Q_n, P_1, \dots, P_n)$$

is the $2n$-tuple of operators of positions and momenta for quantum systems with n degrees of freedom. These operators satisfy the well-known Heisenberg canonical commutation relations (CCR).

As the Weyl–Heisenberg group G_{WH} is a central extension of the additive group $\mathbb{R}^{2n}$, the unitary representations of G_{WH} can be naturally identified with projective representations of $\mathbb{R}^{2n}$. (For more details, see, e.g., Bóna, 2000.) The representation we shall use has its origin in CCR and is given by Weyl operators:

$$U_x = \exp\left(\frac{i}{\hbar} X \cdot S \cdot x\right).$$

Here S is a $2n \times 2n$ matrix such that

$$\begin{array}{ll} S_{ij+n} = -S_{n+jj} = 1, & i = 1, \ldots, 2n, \\ S_{jk} = 0 & \text{otherwise.} \end{array}$$

Now let $\psi \in \mathcal{H}$ be a vector. The corresponding orbit of (the unitary representation of) G_{WH} in $P(\mathcal{H})$ (or, which is the same, the orbit of projective representation of the group $\mathbb{R}^{2n}$) is

$$O_\psi = \{\underline{U_x \psi}; x \in \mathbb{R}^{2n}\}.$$

Now the classical quadratic Hamiltonian will be

$$h(x) = \sum_{j,k=1}^{2n} h_{jk} x_j x_k, \quad h_{jk} = h_{kj} \in \mathbb{R}.$$

The corresponding quantum quadratic Hamiltonian will be

$$H = \sum_{j,k=1}^{2n} h_{jk} X_j X_k$$

where the conditions $h_{jk} = h_{kj}$ imply that this is a symmetric operator.

3. TIME EVOLUTION OF HOMOGENOUS POLYNOMIALS

Let us consider the quantum quadratic Hamiltonian H given above. By a *monomial* of degree m in (operator) variables X_i we mean an expression like $X_{i_1}, X_{i_2}, \ldots, X_{i_m}$. By a *homogenous polynomial* of degree m in variables X_i we mean a linear combination of monomials of degree m. Let us denote by $Y_1, Y_2, \ldots, Y_d$ all the monomials of degree m. Some of them are equal as some pairs from operators X_i commute (e.g. $Q_1 Q_2 P_1$ and $Q_2 Q_1 P_1$ are equal although they are formally different). We shall consider only formally different monomials and it will cause no mistakes in future considerations. (So $d = (2n)^m$.)

Using the derivation property for commutators

$$[X_i, X_j, X_k] = X_i[X_j, X_k] + [X_i, X_k]X_j$$

and CCR it can easily be shown that $[X_i, X_j, X_k]$ depends linearly on (operator) variables X_r or is equal to zero and $[X_i, X_j, X_k, X_l]$ is quadratic in operators X_r

or also equal to zero. More generally, it can be shown that

$$[X_i X_j, X_{k_1} X_{k_2}, \ldots, X_{k_m}] = \sum_{c=1}^{d} c_l Y_l$$

where c_l are some constants. (Again, the result can be zero.) So the commutator of the operator $X_i X_j$ and a monomial of degree m in $X's$ is a homogenous polynomial of degree m (or zero).

The idea of the proof is simple. It is namely

$$\begin{aligned}[X_i X_j, X_{k_1} X_{k_2}, \ldots, X_{k_m}] &= X_{k_i}[X_j, X_{k_1} X_{k_2}, \ldots, X_{k_m}] \\ &\quad + [X_i, X_{k_1} X_{k_2}, \ldots, X_{k_m}] X_j\end{aligned}$$

and using mathematical induction and CCR one can show that $[X_j, X_{k_1} X_{k_2}, \ldots, X_{k_m}]$ is either a homogenous polynomial of degree $m-1$ or zero. It is done by using the equality

$$[X_j, X_{k_1}, \ldots, X_{k_m}] = X_{k_1}[X_j, X_{k_2}, \ldots, X_{k_m}] + [X_j, X_{k_1}] X_{k_2}, \ldots, X_{k_m}.$$

So now we have

Lemma 1. *Let H be a quadratic Hamiltonian, $X_{k_1} X_{k_2}, \ldots, X_{k_m}$ a monomial of degree m. Then*

$$[H, X_{k_1} X_{k_2}, \ldots, X_{k_m}] = \sum_{l=1}^{d} d_l Y_l$$

where d_l are some constants.

Let us now choose one monomial (of degree m) Y_j and consider the corresponding time evolution $Y_j(t)$ in Heisenberg picture for given (quadratic) Hamiltonian H. Let us denote

$$V_t = \exp\left(-\frac{i}{\hbar} t H\right).$$

The corresponding equations for $Y_j(t)$ are then

$$\begin{aligned}\frac{d}{dt} Y_j(t) &= \frac{i}{\hbar}[H, Y_j(t)] = \frac{i}{\hbar}[H, V_{-t} Y_j V_t] = \frac{i}{\hbar} V_{-t}[H, Y_j] V_t \\ &= V_{-t}\left(\sum_{k=1}^{d} c_{jk} Y_k\right) V_t = \sum_{k=1}^{d} c_{jk} Y_k(t)\end{aligned}$$

where c_{jk} are some constants. Here we used Lemma 1. If now $Y(t) = (Y_1(t), \ldots, Y_d(t))$ will be considered to be a column (!) vector of operators, we have an (operator) equation

$$\dot{Y}(t) = CY(t)$$

where C is the corresponding $d \times d$ matrix with matrix elements $C_{jk} = c_{jk}$. The solution has a simple form

$$Y(t) = e^{Ct} Y(0).$$

Let us denote the matrix elements of the matrix e^{Ct} by $[e^{Ct}]_{jk}$. So we proved a

Theorem 2. *Under the notation given above we have*

$$Y_j(t) = \sum_{k=1}^{d} [e^{Ct}]_{jkY_k}$$

The matrix C is uniquely determined by the (quadratic) Hamiltonian H.

This means that the time evolution of a homogenous polynomial of degree m in Heisenberg picture with arbitrary quadratic Hamiltonian is again a homogenous polynomial of degree m. We can say that the set of all homogenous polynomials of degree m is invariant with respect to the time evolution in Heisenberg picture for a quadratic Hamiltonian.

Now let us consider a vector $\psi \in \mathcal{H}$. Let us stay in Heisenberg picture. Then we immediately have

Corollary 3. *Let the quadratic Hamiltonian H be given. Let for some $\psi \in \mathcal{H}$ only the values* $\mathrm{Tr}(P_\psi Y_k)$ *are known* ($k = 1, \ldots, d$). *Then all the time evolved mean values* $\mathrm{Tr}(P_\psi Y_k(t))$ *are uniquely determined by this information and can be explicitly computed by the formula*

$$\mathrm{Tr}(P_\psi Y_k(t)) = \sum_{k=1}^{d} [e^{Ct}]_{jk} \mathrm{Tr}(P_\psi Y_k).$$

By a quantum correlation function of order m we mean the function dependent on $\psi \in \mathcal{H}$ and defined by the expression

$$c(\psi) = \mathrm{Tr}(P_\psi (X_{k_1} - x_{k_1} I)(X_{k_2} - x_{k_2} I) \cdots (X_{k_m} - x_{k_m} I)).$$

Here I denotes the identity operator and

$$x_j = \mathrm{Tr}(P_\psi X_j)$$

are just the corresponding mean values of observables X_j. Let us consider the corresponding time evolution of this correlation function

$$c(\psi(t)) = \mathrm{Tr}(P_{\psi(t)} (X_{k_1} - x_{k_1}(t)) \cdots (X_{k_m} - x_{k_m}(t)))$$

Here

$$\psi(t) = V_t \psi$$

be the time evolved state in Schrodinger picture and

$$x_j(t) = \mathrm{Tr}(P_{\psi(t)} X_j)$$

are the corresponding time evolved mean values. From the Ehrenfest theorem (see, e.g., Messiah, 1978) now immediately follows that if we know $x_j(0)$ then $x_j(t)$ can be computed from classical Hamiltonian equation for corresponding classical quadratic Hamiltonian.

In Heisenberg picture we can write

$$c(\psi(t)) = \mathrm{Tr}(P_\psi(X_{k_1}(t) - x_{k_1}(t)) \cdots (X_{k_m}(t) - x_{k_m}(t))).$$

Now a simple computation gives

$$\begin{aligned} &(X_{k_1}(t) - x_{k_1}(t)) \cdots (X_{k_m}(t) - x_{k_m}(t)) \\ &= \sum_{(l_1,\ldots,l_p,n_1,\ldots,n_r)} (-1)^p x_{l_1}(t) \cdots x_{l_p}(t) X_{n_1}(t) \cdots X_{n_r}(t) \end{aligned}$$

where $(l_1, \ldots, l_p, n_1, \ldots, n_r)$ is a partition of the index m-tuple $(k_1, \ldots, k_m)$. Having in mind that all the numbers $x_j(t)$ are known if the values $\mathrm{Tr}(P_\psi X_j)$ are known, it follows that $c(\psi(t))$ can be computed if we know all the corresponding values

$$\mathrm{Tr}(P_\psi X_{n_1}(t) \cdots X_{n_r}(t)).$$

But

$$X_{n_1}(t) \cdots X_{n_r}(t) = V_{-t}(X_{n_1} \cdots X_{n_r}) V_t = Y_k(t)$$

for some monomial Y_k of order r. Then according to the Corollary 3 the expression

$$\mathrm{Tr}(P_\psi X_{n_1}(t) \cdots X_{n_r}(t)) = \mathrm{Tr}(P_\psi Y_k(t))$$

can be explicitly computed as a (time dependent) linear combination of values $\mathrm{Tr}(P_\psi Z_j)$ where the operators Z_j are just all monomials of degree r. Recall that $x_j(t)$ is determined by $\mathrm{Tr}(P_\psi X_j)$ and X_j is a monomial of degree 1. Then we have proved the following

Corollary 4. *Let $\psi \in \mathcal{H}$ and W_j are all possible monomials of degrees $1, \ldots, m$. Let all the values $Tr(P_\psi W_j)$ are known. Then the time evolution of arbitrary quantum correlation function of order m*

$$c(\psi(t)) = Tr(P_\psi(X_{k_1}(t) - x_{k_1}(t)) \cdots (X_{k_m}(t) - x_{k_m}(t)))$$

can be computed explicitly in terms of linear combinations of the given values $Tr(P_\psi W_j)$ *with time-dependent coefficients.*

Remark. The dispersion of the quantum mechanical observable X_j in the state ψ is $\mathrm{Tr}(P_\psi(X_j - x_j)^2)$ which is a quantum correlation function of second order.

If we know all the mean values of the monomials of degrees 1, 2 in the state ψ then the Corollary 4 implies that we know also the time evolved values of the dispersion. Similar statements are true also for higher centered moments of quantum observables X_j.

Remark. As an example of computation of time evolution of dispersion (for the free particle) see Lemma 8.

Now from Theorem 2 we have the time evolution of X_j in Heisenberg picture. There exist some numbers $c_{j1}(t), \ldots c_{j\,2n}(t)$ such that

$$X_j(t) = c_{j1}(t)X_1 + \cdots + c_{j\,2n}(t)X_{2n}. \tag{1}$$

Let

$$V_t = \exp\left(-\frac{i}{\hbar}tH\right)$$

and U_x are the Weyl operators defined above. We shall prove

Theorem 5. *Under the notation given above it is*

$$V_t U_x (V_t)^{-1} = U_{x(t)}$$

where $x(t)$ *is a classically evolved state for the Hamiltonian* h *in time* t *if the initial condition is* $x(0) = x$.

Proof: It is namely

$$V_t U_x (V_t)^{-1} = V_t \exp\left(\frac{i}{\hbar}X_j S_{jk} x_k\right)(V_t)^{-1} = \exp\left(\frac{i}{\hbar}V_t X_j (V_t)^{-1} S_{jk} x_k\right).$$

From (1) it follows

$$V_t X_j (V_t)^{-1} = c_{j1}(-t)X_1 + \cdots + c_{j\,2n}(-t)X_{2n}.$$

In this way, numbers $x_k(t)$ are determined determining the vector

$$x(t) = (x_1(t), \ldots, x_{2n}(t)),$$

such that it holds

$$V_t U_x (V_t)^{-1} = \exp\left(\frac{i}{\hbar}X_j S_{jk} x_k(t)\right) = U_{x(t)}.$$

From the Ehrenfest theorem we obtain that $x(t) = y(t)$, where $y(t)$ is the classically evolved state for Hamiltonian h in time t if $y(0) = x$. It is namely

$$V_t U_x = U_{x(t)} V_t,$$

so

$$V_t U_x \phi = U_{x(t)} V_t \phi, \tag{2}$$

where ϕ is a generating vector of some orbit of G_{WH}, so that the following conditions are satisfied:

$$\mathrm{Tr}(P_\phi X_j) = 0, \quad j = 1, \ldots, 2n.$$

Let us consider the mean values of observables X_j for both left and right sides of (2). For the left side we have from Ehrenfest theorem that these mean values are exactly $y_j(t)$ where $y(t)$ is defined above. From the relation (1) it follows that the vector $V_t\phi$ is also a generating vector of some orbit of G_{WH}. Now it immediately follows that for the right side the considered mean values are equal to $x_j(t)$, so we have $y(t) = x(t)$. □

Let us now recall that O_ϕ is an orbit of G_{WH} constructed from a vector $\phi \in \mathcal{H}$. We now have

Corollary 6. (see Kubisz, 1992; Polakovič, 2001c) *Let $\phi \in \mathcal{H}$ and O_ϕ is the corresponding orbit, let $\psi_1 \in \mathcal{H}$, $\psi_2 \in \mathcal{H}$ be such that $\underline{\psi_1} \in O_\phi$, $\underline{\psi_2} \in O_\phi$. Let $\underline{\psi_1(t)} = V_t\psi_1$, $\psi_2(t) = V_t\psi_2$. If now $\phi' \in \mathcal{H}$ is such that $\underline{\psi_1(t)} \in O_{\phi'}$ then also $\underline{\psi_2(t)} \in O_{\phi'}$.*

Remark. This means that two initial states belong to a single orbit of G_{WH} then also the corresponding time evolved states (for the same time t and quadratic Hamiltonian H) belong to a single orbit of G_{WH}. An alternative formulation of the Corollary 6 could be that the time evolution V_t defines a bijection between the orbits O_ϕ and $O_{\phi'}$.

4. FREE PARTICLE AND ORBITS OF WEYL–HEISENBERG GROUP

Let us now consider a special case of quadratic Hamiltonian, namely the quantum free particle in three dimensions. So the Hamiltonian will be

$$H = \frac{P^2}{2m} = \frac{P_1^2 + P_2^2 + P_3^2}{2m}.$$

The corresponding time evolution will be given by operator

$$V_t = \exp\left(-\frac{i}{\hbar} t H\right) = \exp\left(-\frac{i}{\hbar} t \frac{P^2}{2m}\right).$$

As the number of degrees of freedom is 3, we consider the corresponding version of G_{WH}, namely the central extension of the additive group $\mathbb{R}^6$. As we know, the corresponding orbits $O_\psi = O^1_\psi$, $\psi \in \mathcal{H}$ will be six-dimensional with canonical coordinates

$$x = (q_1, q_2, q_3, p_1, p_2, p_3)$$

where

$$q_i = \mathrm{Tr}\,(P_{\phi_x} Q_i), \qquad p_i = \mathrm{Tr}\,(P_{\phi_x} P_i) \quad i = 1, 2, 3$$

where $\phi_x \in \mathcal{H}$, P_{ϕ_x} is the corresponding (uniquely determined) element of the orbit O_ψ. Let now at time $t = 0$ the particle is on the orbit O_ψ in state ϕ_x where

$$x = (q_1', q_2', q_3', p_1', p_2', p_3')$$

is arbitrary. Let at time $t = T$ it is again on the orbit O_ψ in state ϕ_y where

$$y = (q_1'', q_2'', q_3'', p_1'', p_2'', p_3'').$$

Now from Ehrenfest theorem and the momentum conservation law for classical free particle it is $p_1' = p_1''$, $p_2' = p_2''$, $p_3' = p_3''$. Let us denote $q_1 = q_1'' - q_1'$, $q_2 = q_2'' - q_2$, $q_3 = q_3'' - q_3'$. Let us also denote

$$q = (q_1, q_2, q_3, 0, 0, 0)$$

so it is

$$U_q = \exp\left(\frac{i}{\hbar}(q_1 P_1 + q_2 P_2 + q_3 P_3)\right).$$

It is clearly

$$\underline{\phi_y} = \underline{U_q \phi_x}.$$

But at the same time

$$\underline{\phi_y} = \underline{V_T \phi_x}.$$

Now we have

$$\begin{aligned} \underline{V_{2T}\phi_x} = \underline{V_T V_T \phi_x} &= \underline{V_T U_q \phi_x} \text{ (from commutativity of } V_T, U_q) \\ &= \underline{U_q V_T \phi_x} = \underline{U_q U_q \phi_x} = \underline{U_{2q}\phi_x} \end{aligned}$$

where we denoted for $n \in \mathbb{N}$

$$nq = (nq_1, nq_2, nq_3, 0, 0, 0).$$

So by mathematical induction we immediately obtain

$$\underline{V_{nT}\phi_x} = \underline{U_{nq}\phi_x}.$$

for all $n \in \mathbb{N}$. So we proved

Lemma 7. *Let O_ψ be arbitrary six-dimensional orbit of G_{WH}, $\underline{\phi} \in O_\psi$ is arbitrary. Let for some $T > 0$ also $\underline{V_T\phi} \in O_\psi$. Then for all $n \in \mathbb{N}$ it is $\underline{V_{nT}\phi} \in O_\psi$.*

Remark. So we have some periodic structure for points of intersection of the time evolution of free particle with the given orbit O_ψ. The periodicity can be considered "in time and space." The time period is T, the "space" period (in canonical coordinates on O_ψ) is given by vector (translation)

$$q = (q_1, q_2, q_3, 0, 0, 0)$$

Let now $\phi(0) \in \mathcal{H}$ is the initial state of the system (free particle) and let

$$q_j(0) = \mathrm{Tr}\,(P_{\phi(0)}Q_j), \qquad p_j(0) = \mathrm{Tr}\,(P_{\phi(0)}P_j), \quad j = 1, 2, 3.$$

The time evolution is given by

$$\phi(t) = V_t\phi(0).$$

Let us denote

$$q_j(t) = \mathrm{Tr}\,(P_{\phi(t)}Q_j), \qquad p_j(t) = \mathrm{Tr}\,(P_{\phi(t)}P_j), \quad j = 1, 2, 3$$

the mean values of positions and momenta evolved in time t. We shall now consider the time evolution of dispersion of the position Q_j (j is arbitrary). It is

$$\mathrm{Tr}\,(P_{\phi(t)}(Q_j - q_j(t))^2) = \mathrm{Tr}\left(P_{\phi(t)}Q_j^2\right) - q_j(t)^2.$$

It is easy to compute the numbers $q_j(t)^2$ as according to the Ehrenfest theorem the values $q_i(t)$, $p_i(t)$ have classical time evolution because the Hamiltonian is quadratic. So we have

$$q_j(t) = q_j(0) + \frac{p_j(0)}{m}t.$$

It is

$$\begin{aligned}\mathrm{Tr}\left(P_{\phi(t)}Q_j^2\right) &= \mathrm{Tr}\big(V_t P_{\phi(0)} V_{-t} Q_j^2\big) = \mathrm{Tr}\left(P_{\phi(0)} V_{-t} Q_j^2 V_t\right) \\ &= \mathrm{Tr}\,(P_{\phi(0)} V_{-t} Q_j V_t V_{-t} Q_j V_t).\end{aligned}$$

So now we shall compute the time evolution of Q_j in Heisenberg picture

$$Q_j(t) = V_{-t}Q_jV_t.$$

It is

$$\begin{aligned}&\frac{d}{dt}\bigg|_{t=s} Q_j(t) = \frac{d}{dt}\bigg|_{t=0} (V_{-t}Q_j(s)V_t) = \frac{i}{\hbar}\left[\frac{P^2}{2m}, Q_j(s)\right] = \\ &= \frac{i}{2\hbar m}[P^2, V_{-s}Q_jV_s] = \frac{i}{2\hbar m}[V_{-s}P^2V_s, V_{-s}Q_jV_s] = \frac{i}{2\hbar m}V_{-s}[P^2, Q_j]V_s \\ &= \frac{1}{m}V_{-s}P_jV_s = \frac{1}{m}P_j.\end{aligned}$$

So we obtained a differential equation for $Q_j(t)$, the solution of which is

$$Q_j(t) = Q_j + \frac{1}{m} P_j t.$$

It is formally similar to the corresponding expression for $q_j(t)$. So

$$\begin{aligned}
\mathrm{Tr}\left(P_{\phi(t)} Q_j^2\right) &= \mathrm{Tr}\left(P_{\phi(0)}\left(Q_j + \frac{1}{m} P_j t\right)^2\right) \\
&= \mathrm{Tr}\left(P_{\phi(0)}\left(Q_j^2 + \frac{t}{m}(Q_j P_j + P_j Q_j) + \frac{t^2}{m^2} P_j^2\right)\right) \\
&= \mathrm{Tr}\left(P_{\phi(0)} Q_j^2\right) + \frac{t}{m}\mathrm{Tr}\left(P_{\phi(0)}(Q_j P_j + P_j Q_j)\right) + \frac{t^2}{m^2}\mathrm{Tr}\left(P_{\phi(0)} P_j^2\right) \\
&= a + bt + ct^2.
\end{aligned}$$

As

$$\begin{aligned}
q_j(t)^2 = \left(q_j(0) + \frac{p_j(0)}{m} t\right)^2 &= q_j(0)^2 + 2q_j(0)\frac{p_j(0)}{m} t + \frac{p_j(0)^2}{m^2} t^2 \\
&= a' + b't + c't^2,
\end{aligned}$$

the considered (time evolved) dispersion is

$$\mathrm{Tr}\left(P_{\phi(t)} Q_j^2\right) - q_j(t)^2 = (a - a') + (b - b')t + (c - c')t^2 = \tilde{a} + \tilde{b}t + \tilde{c}t^2.$$

Here

$$\begin{aligned}
\tilde{a} &= \mathrm{Tr}\left(P_{\phi(0)} Q_j^2\right) - \mathrm{Tr}\left(P_{\phi(0)} Q_j\right)^2, \\
\tilde{b} &= \frac{1}{m}\left(\mathrm{Tr}\left(P_{\phi(0)}(Q_j P_j + P_j Q_j)\right) - 2\mathrm{Tr}\left(P_{\phi(0)} Q_j\right)\mathrm{Tr}\left(P_{\phi(0)} P_j\right)\right), \\
\tilde{c} &= \frac{1}{m^2}\left(\mathrm{Tr}\left(P_{\phi(0)} P_j^2\right) - \mathrm{Tr}\left(P_{\phi(0)} P_j\right)^2\right).
\end{aligned}$$

As the spectrum of the operator P_j is purely continuous, $\phi(0)$ cannot be an eigenvector of P_j, so it is $\tilde{c} \neq 0$. As the dispersion is always nonnegative, it is for all $t \in \mathbb{R}$

$$\tilde{a} + \tilde{b}t + \tilde{c}t^2 \geq 0,$$

so we necessarily have $\tilde{c} > 0$. We have proved.

Lemma 8. *The time evolution (for time t) of the dispersion of the observable Q_j for the free particle is given by an expression $\tilde{a} + \tilde{b}t + \tilde{c}t^2$ where $\tilde{a}, \tilde{b}, \tilde{c}$ are the corresponding real constants, $\tilde{c} > 0$.*

Remark. The precise form of the constants $\tilde{a}, \tilde{b}, \tilde{c}$ is given above. It is an example of computation of time evolution of some dispersion which was mentioned in the second Remark after Corollary 4.

Remark. This result confirms a well-known fact from QM, namely the "spreading of the wave packet" for the free particle.

Now we formulate a

Lemma 9. (see Bóna, 1984, Note 4.1.7(i)) *Let O_ψ be an orbit of G_{WH}. Let $P_\phi \in O_\psi$. Then the quantum correlation functions are constant on O_ψ, i.e. the value*

$$c(\phi) = Tr(P_\phi(X_{j_1} - x_{j_1}I)(X_{j_2} - x_{j_2}I)\cdots(X_{jm} - x_{jm}I))$$

does not depend on the choice $P_\phi \in O_\psi$. (Here $x_i = Tr(P_\phi X_i)$.)

Now we are ready to prove the following

Theorem 10. *Let the orbit O_ψ be given and $\phi \in \mathcal{H}$ is such that $P_\phi \in O_\psi$. Let $\phi(t) = V_t\phi$ be the time evolved state (of the free particle). Then for $t \neq 0$ we have $P_{\phi(t)} \notin O_\psi$.*

Proof: Let us suppose that $P_{\phi(T)} \in O_\psi$ for some $T > 0$ (the case $T < 0$ is analogous). From Lemma 7 we have $P_{\phi(nT)} \in O_\psi$ for all $n \in \mathbb{N}$. So for arbitrary $t_0 \in \mathbb{R}$ there exists $t > t_0$ such that $P_{\phi(t)} \in O_\psi$. From the Lemma 9 now it follows that the dispersion of Q_j in the state $P_{\phi(t)}$ is the constant number determined by O_ψ. But this is in contradiction to the Lemma 8. □

Remark. This Theorem says that if the initial state for the free particle is from the orbit O_ψ then the time evolution never more intersects this orbit.

Now using the fact that the time evolution of the dispersion of Q_j is given by the expression $\tilde{a} + \tilde{b}t + \tilde{c}t^2$ and $\tilde{c} > 0$, we have that for arbitrary

$$A > \min\{\tilde{a} + \tilde{b}t + \tilde{c}t^2; t \in \mathbb{R}\}$$

there exist numbers $t_1 \neq t_2$ such that

$$\tilde{a} + \tilde{b}t_1 + \tilde{c}t_1^2 = \tilde{a} + \tilde{b}t_2 + \tilde{c}t_2^2 = A.$$

Let the corresponding orbits O_{ψ_1}, O_{ψ_2} are such that $P_{\phi(t_1)} \in O_{\psi_1}, P_{\phi(t_2)} \in O_{\psi_2}$. From Theorem 10 we have that $O_{\psi_1} \cap O_{\psi_2} = \emptyset$. So we have

Corollary 11. *There exist a number $A_0 \in \mathbb{R}$ such that for all $A > A_0$ there always exist two different orbits O_{ψ_1}, O_{ψ_2} such that both of them have the same (constant) dispersion A of the position Q_j.*

Remark. This result gives a partial answer to the natural question: knowing that the dispersions of all X_j are constants on the orbits of G_{WH}, are there some different orbits which have the same dispersion for some chosen X_j?

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REFERENCES

Bóna, P. (1984). *Classical Projections and Macroscopic Limits of Quantum Mechanical Systems*, unpublished monograph, Bratislava.

Bóna, P. (2000). Extended quantum mechanics, *Acta Physica Slovaca* **50**(1), 1–198.

Kubisz, P. (1992). *Dynamics of Coherent States for Quadratic Hamiltonians*, Diploma thesis, Comenius University, Bratislava, Slovak.

Marsden, J. E. and Ratiu, T. S. (1999). *Introduction to Mechanics and Symmetry*, second edition, Springer, New York.

Messiah, A. (1978). *Quantum Mechanics I*, Nauka, Moscow (Russian).

Perelomov, A. M. (1986). *Generalized Coherent States and Their Applications*, Springer, Berlin.

Polakovič, M. (1998). Limit of classical projections of quantum mechanics as $h \to 0$, *International Journal of Theoretical Physics* **37**(12), 2923–2934.

Polakovič, M. (2000). Some properties of systems with quadratic Hamiltonions, *Journal of Electrical Engineering* **51**(12/s), 63–67.

Polakovič, M. (2001a). Erratum: Limit of classical projections of quantum mechanics as $h \to 0$, *International Journal of Theoretical Physics* **40**(3), 767.

Polakovič, M. (2001b). Classical projections of quantum mechanics and the limit $h \to 0$, *International Journal of Theoretical Physics* **40**(3), 755–765.

Polakovič, M. (2001c). *Time Evolution of Quantum Systems with Quadratic Hamiltonians and Orbits of Weyl–Heisenberg Group*, 3rd Workshop on Functional Analysis and its Applications, 4–9 September, 2001, Nemecká, Slovakia, pp. 62–63.

Weyl, H. (1931). *The Theory of Groups and Quantum Mechanics*, Dover, New York.

Želobenko, D. P. and Stren, A. I. (1983). *Lie Group Representation*, Nauka, Moscow (Russian).